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Numerical Computation of the Magnitude and Frequency of the Lift on a Circular Cylinder

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NUMERICAL COMPUTATION OF THE MAGNITUDE AND FREQUENCY OF THE LIFT ON A CIRCULAR CYLINDER

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CONTENTS

The shear layers bounding the formation region behind a bluff body are modelled by individual elementary vortices. These vortices are free to move under the action of the flow past the body and the velocity field of the vortices. This configuration generates a vortex street. The strengths of the elementary vortices are determined from the velocity calculated at the point at which the vortices are caused to appear. It is essential that this point be downstream of the boundary layer separation point and also that it is allowed to oscillate. The determination of the strengths of the vortices is a crucial point in the calculation and is performed by a careful examination of the flow in the formation region. The basis of this work is to be found in the author's earlier paper (1966). The independent variables of the model are reduced to a minimum prior to calculation.

Calculations have been performed, on a digital computer, with the independent variables corresponding to two flows. At a high subcritical Reynolds number the values of the oscillating velocities, lift force and the scale of the formation region generated by the model agree with experiment. The frequency generated differs from that observed in the way predicted by an assessment of the effect of ignoring turbulent and viscous effects.

In the second series of calculations the variables were matched to the flow observed at a Reynolds number of 2000 in the absence of free-stream disturbances. The diminutive values of oscillating quantities observed under these conditions are also generated by the model.

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1. Introduction

This paper is the sequel of the author's two previous publications (1965, 1966). The first of these emphasizes the large range of amplitude of the lift observed when the disturbance level of the free stream is very small. The second lays the physical foundation of the present work by proposing a theory of the mechanism which determines the frequency of vortex shedding.

Despite numerous measurements of the now well defined frequency of vortex shedding from a circular cylinder the explanation of the frequency determining mechanism has withstood attack. The recognition of the importance of the near wake or formation region of the vortices seems to have dawned at least as long ago as 1927 when Fage & Johansen performed their experiments on this flow. Nevertheless the formation region is a part of the flow avoided by the majority of the experimenters who have since undertaken the observation of the vortex wake with the hot-wire anemometer. The interpretation of signals from hot wires located in the formation region is very difficult. The early measurements of Fage & Johansen (1927) and Schiller & Linke (1933) stood alone until recently when interest has revived: see, for example, Bloor (1964) and Hanson & Richardson (1965).

Fage & Johansen's (1927) work formed the basis of later work by Roshko (1954) and we shall also take this as our starting point in the present treatment. The part of their work of vital importance here is the determination of the average rate of shedding of circulation from the vicinity of one side of the cylinder. Consideration of the formation region in the case of a circular cylinder is complicated by the oscillatory motion of the separation points: even so the answer to the question of what determines the frequency of vortex shedding was not forthcoming for flat plates or other bluff bodies with salient edges at which the separation points are anchored. The problem is essentially the same as that outstanding in unsteady aerofoil theory in which the reciprocal action of the wake on the boundary layer and vice versa is difficult to formulate in an analytical treatment: see, for example, Sears (1956).

In the present treatment the shear layers which spring from the sides of the body are replaced by ideal fluid point vortices somewhat in the manner in which Abernathy & Kronauer (1962) dealt with the formation of an infinite vortex street. The elementary vortices are constrained to move under the action of the potential flow. New vortices arise in the formation region in a manner which will be discussed at length below but which is vitally important for a realistic representation of the actual flow. The number of disposable variables is reduced to a minimum on the basis of physical argument and in the same way the effects of varying Reynolds number are incorporated. The basis of this argumentation is to be found in the author's earlier paper (1966). It seems likely that a similar treatment for bluff bodies other than the circular cylinder could be undertaken by conformal transformation of the flow into that about the circle.

Similar but more sophisticated work was done by Fromm & Harlow (1963). They present numerical solutions of the equations of motion of a viscous incompressible flow past bluff bodies started impulsively from rest. They have demonstrated that the filament line patterns agree with flow visualization observations. This powerful method has great potentialities though as yet has not been programmed to give the forces on the body;

no explanation of the details of the formation of vortices has yet been forthcoming. The present paper describes a considerably simplified investigation along parallel lines. By simplification it was hoped to separate the important factors and so gain a physical insight into the mechanics of the flow. Many of the physical ideas presented in the previous paper (Gerrard 1966) evolved during the work now to be presented.

2. The method in outline

The shear layers springing from the sides of the cylinder will be represented by discrete vortices which will appear at regular intervals. Starting from some initial asymmetrical vortex distribution representing the wake of the cylinder the elementary vortices will be allowed to move under the action of the velocity field. The motion will sooner or later become independent of the initial conditions. The velocity distribution of the vortex of strength K will be taken to be $K/2\pi r$, where r is the distance from the centre of the vortex. The method involves a two-dimensional potential flow calculation in which the effects of viscosity and transition to turbulence may be simulated. As yet this has not been done except in an average manner by the variation of a disposable parameter describing the strength of the individual vortices.

A particularly important part of the vortex model of the flow and the most difficult aspect of the design of the model, is the positioning of the points of appearance of the elementary vortices representing the vortex sheets and the determination of their strength. The difficulty of treatment of the motion of the separation points is circumvented by considering the discharge of circulation across a control surface situated downstream of separation. This will be discussed in the next section. It will be attempted to simulate the effect of variation of Reynolds number by variation of the distance of the points of vortex appearance from the wake axis. The effects of turbulence and viscosity on the thickness of the shear layer are expected to be of major importance (see Gerrard 1966) but these have as yet not been fully incorporated in the model. A comparison of the results with experiment is, however, instructive even with this omission.

The model of the flow to be presented involves considerable computing time. Many simpler models have been investigated but these suffer from an abundance of disposable parameters the choice of which is difficult to justify. At each step of the calculation the velocity is determined at each vortex position, the vortices are moved in accordance with these velocities and the time interval of each step: two new elementary vortices are introduced near the cylinder; the lift, drag and other quantities are calculated and the process is then repeated. This simple but lengthy procedure was performed on the Pennsylvania State University I.B.M. 7074 computer.

3. The potential flow vortex model of the wake

The wake of a bluff body may be represented by a potential flow model which contains only vortices provided the effects of viscosity and turbulence can be adequately incorporated. The long standing determination (see Goldstein 1938) of the drag due to a Kármán vortex street shows that the mean forces may be computed in this way. A consideration of the oscillating properties of the flow which involves the wake close behind the body presents a problem of greater magnitude.

In a recent paper the author (1966) proposed a theory of the mechanism of vortex shedding which includes a treatment of the manner in which the frequency of the oscillations is determined. In that paper it is proposed that there are two characteristics of the flow which are of major importance: one is the drawing across the wake, by the growing vortex, of the shear layer shed from the other side of the body; this cuts off the further supply of circulation to the growing vortex which is then shed. The other feature of the flow of major importance is the extent to which the shear layer has diffused by the time it reaches the end of the formation region where it is drawn across the wake. The first of these features would be expected to be automatically included in the vortex model of the wake; the second will not be automatically included. Rosenhead (1931) has shown by numerical computation that a simple shear layer rolls up into vortices and Bloor (1964) has observed this in the shear layers bounding the formation region behind a circular cylinder. This rolling up is only part of the diffusion process, turbulence in the shear layer will produce enhanced diffusion to a significant extent. Entrainment of fluid from the interior of the formation region particularly into the turbulent part of the shear layer has not been modelled. Diffusing vortices and the possibility of entrainment are the obvious next stages in a computation along the lines developed here.

3·1. *Method of calculation*

At various stages in the work programs have been altered in detail; the basic method of calculation will be presented here. In an earlier paper (Gerrard 1963) a similar method was outlined in which the elementary vortices were caused to appear near the cylinder with their strengths independently determined and oscillating with the Strouhal frequency. In the present model the strength and frequency are dependent variables determined by

TABLE 1

x is the streamwise coordinate,
y the coordinate perpendicular to the free-stream direction.
Origin of coordinates at the cylinder centre.

$$r^2 = x^2+y^2, \quad r_1^2 = x_1^2+y_1^2.$$

Distances non dimensionalized with the cylinder radius, a.
Velocities non dimensionalized with the free-stream speed, U.
H = vortex strength; non dimensionalized with $2\pi Ua$.
The axes are right-handed and H is positive if counterclockwise.
Velocity at x, y due to potential flow past cylinder;

$$u = 1-(x^2-y^2)/r^4, \quad v = -2xy/r^4.$$

Velocity at x, y due to vortex of strength H at x_1, y_1;

$$u = H(y_1-y)/(r^2+r_1^2-2xx_1-2yy_1),$$
$$v = H(x-x_1)/(r^2+r_1^2-2xx_1-2yy_1).$$

Velocity at x, y due to the image of the vortex above:

$$u = -H(y_1-r_1^2y)/(1+r^2r_1^2-2xx_1-2yy_1),$$
$$v = H(x_1-r_1^2x)/(1+r^2r_1^2-2xx_1-2yy_1).$$

Lift and rag coefficients due to vortex of strength H at x, y having velocity u, v.
'Steady' part (due to $\frac{1}{2}\rho(u^2+v^2)$:

$$C_D = -2\pi Hv, \quad C_L = 2\pi Hu.$$

'Unsteady' part (due to $\partial\phi/\partial t$; $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$):

$$C_D = 2\pi H\{v(x^2-y^2)-2uxy\}/r^4, \quad C_L = 2\pi H\{u(x^2-y^2)+2vxy\}/r^4.$$

the progress of the model of the flow. Initially a few vortices (usually three) are assigned positions and strengths. This arrangement represents the vortex wake and its asymmetry causes the oscillations to commence. The x coordinate (see table 1) of the position at which the next vortices are to appear are also assigned and fixed: The y coordinate of the position of appearance is only partly determined independently. The velocity at each actual and nascent vortex position is calculated assuming potential flow past the cylinder, the inverse distance relation for the velocity field of the vortices and of their images at the inverse points within the body. (There are not images at the centre of the cylinder because the vortices have been shed from it and leave circulation opposite to their own on the body.) The velocities S, T and V (defined in figures 2 and 3), on the cylinder

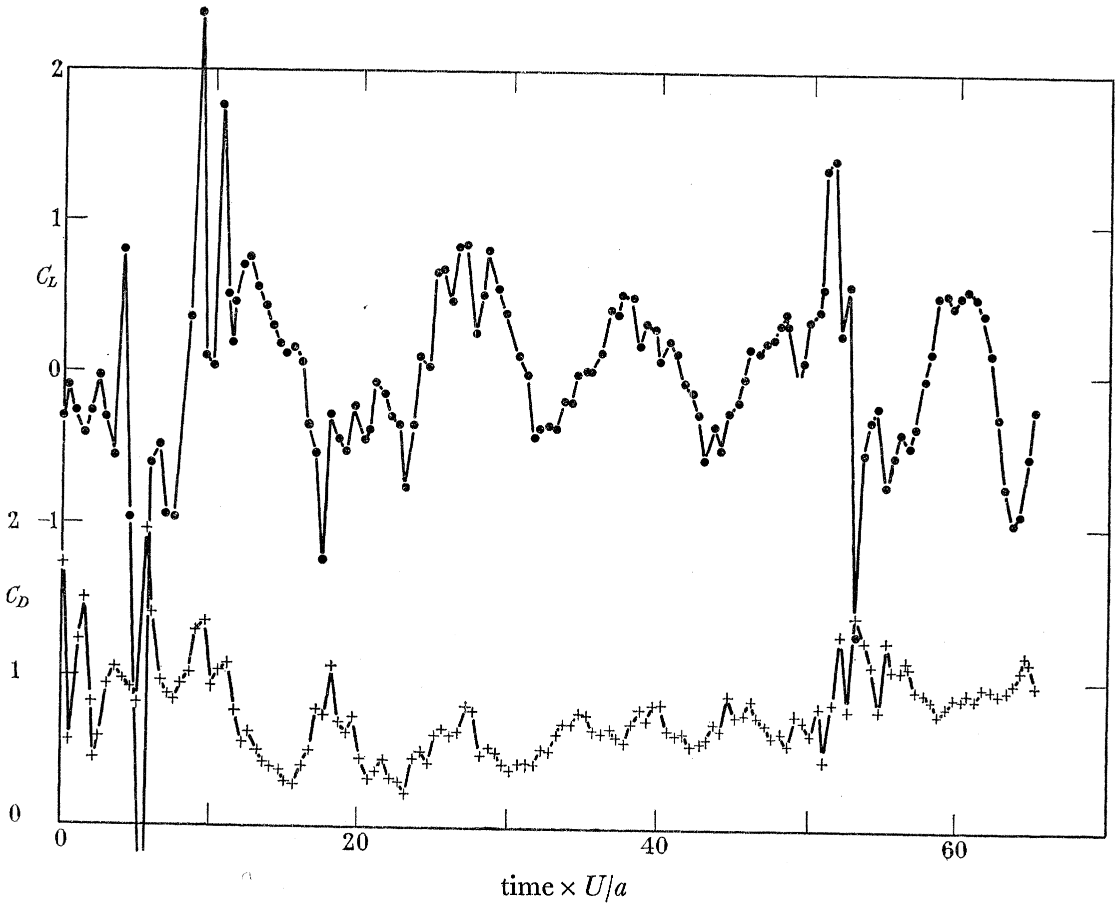


FIGURE 1. Computed lift and drag coefficients as functions of time. $d' = 1{\cdot}1$, mean wake semi-width $= 1{\cdot}25$, $k = 0{\cdot}5$, $R = 0{\cdot}5$, $Z = 0{\cdot}3$, time interval $= 0{\cdot}5a/U$. Coalescence of near vortices is included. Reverse flow vortices are entrained without change in strength.

surface are also calculated. The vortices are now moved to new positions by convection, with the velocity determined at their present position, for a predetermined small time interval. New vortices are placed at the positions of appearance with strengths determined as discussed below. At this stage the vortices have corresponding velocities which are those with which they arrive at their assigned positions. For the calculation of C_L and C_D, the lift and drag coefficients the nascent vortices are given a velocity equal to that of the vortex which last left the position of appearance. In this case, however, only the streamwise component of the speed is assigned: The reasons for this will be discussed

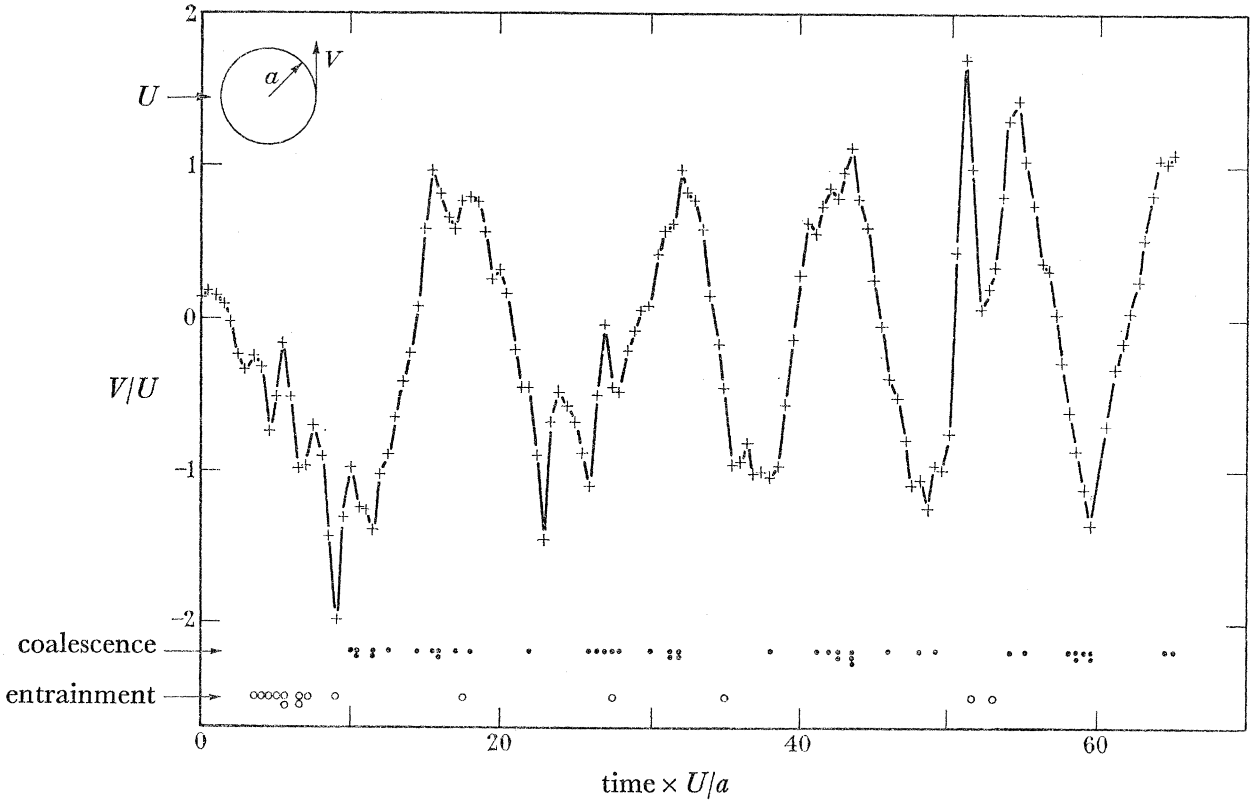


FIGURE 2. Waveform of the velocity at the rear of the cylinder. Conditions as in figure 1. The times of coalescence and entrainment are indicated.

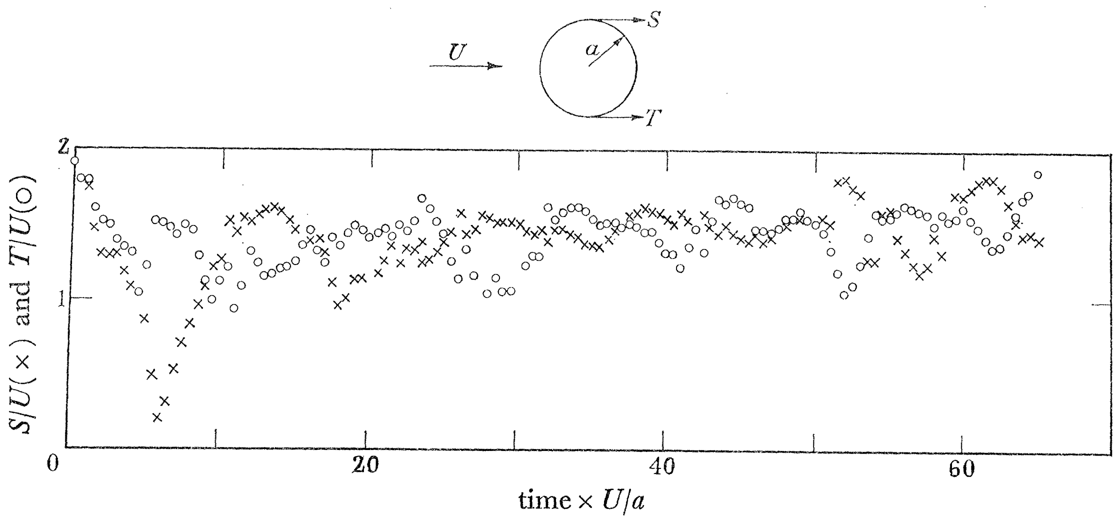


FIGURE 3. Velocities at the shoulders of the cylinder. Conditions as in figure 1.

below. The lift and drag are calculated exactly as described by Sarpkaya (1963). We note that he includes an image at the centre of the circle which of course is omitted here. The equations used in the calculation are shown in table 1.

The whole process is now repeated many times with two more vortices in the wake at each repetition. When desired the position of the vortices are available. The details of the model and method of calculation will now be discussed.

3·2. *Details of the method*

3·2·1. *The position of appearance of the vortices representing the vortex sheet*

It has been considered essential to isolate the shear layer from the separation point in order to determine the rate of shedding of circulation from one side of the body. One reason for this is that the separation point oscillates. Even if the separation point were fixed there are other difficulties which arise in a determination of the convection of circulation across a plane close to the body.

The general idea behind the determination of vortex strength is as follows. Suppose that at any time more circulation has been shed from one side, say the top, of the cylinder than the other. The velocity produced in the region of the cylinder will be such that there will tend to be less circulation shed from the top of the cylinder than from the bottom in the next instant of the motion. This will presumably continue for some time and may overshoot. The possibility of an oscillation in the rate of shedding of circulation is obviously present even without an oscillating wake. One cannot say, a priori, that the oscillation will settle down to a constant amplitude: it may die out or may increase with time. In fact both of these characteristics have been produced by suitable, if unrealistic, choice of parameters.

When one tries to apply these ideas in detail difficulty is encountered. Suppose one calculates the velocity at a supposedly fixed separation point (S say, in figure 3) and assumes that this is the velocity outside the boundary layer at this point. The rate of shedding of circulation will be approximately $\frac{1}{2}S^2$. This velocity is determined principally by the images of vortices within the cylinder. Each elementary vortex, from one side, has roughly equal weight in determining S if their strengths are equal. The images of distant vortices, however, have little effect because they are close to oppositely signed images of vortices shed from the other side of the body. The variation in the speed S is therefore strongly governed by vortices in close proximity. In the approximate treatment undertaken here models based on these ideas produced unsatisfactory results; large values of S^2 were generated from which the model did not recover. This difficulty is presumably encountered because the determination of vortex strength in this way is equivalent to the model 'pulling itself up by its own shoelaces.'

For the above reasons and because of the difficulty associated with a moving separation point, a control surface was chosen downstream of separation. The rate of convection of circulation across the control surface depends on the velocities on both sides of the shear layer. These velocities again depend predominantly on vortices close to the control plane. It appears to be impossible to use these speeds to determine the strength to be assigned to the vortex in that position. The velocity of convection both in the direction of the shear layer and at right angles to this, on the other hand, depend on the vortices far removed from this location. This is true provided the curvature of the layer is not large at the control surface and provided also that the surface is sufficiently far from the body. If the control surface is too far from the body, neglecting the vortex sheets upstream of it does not produce a sufficiently close first approximation. The control surface, which is perpendicular to the free-stream direction, also must not be too far from the centre of the body compared with the length of the formation region. The position chosen for the control surface was one cylinder radius downstream of the centre of the cylinder. The strength of the nascent vortex appearing at the control surface has been determined from the velocity computed at the position of the nascent vortex. The effects of ignoring the vortex sheet upstream of the control surface will be assessed and to some extent corrected for.

3·2·2. *The strengths of the elementary vortices*

The rate of convection of circulation in a plane vortex sheet separating uniform flows with speeds u_1 and u_2 is $\frac{1}{2}(u_2^2-u_1^2)$. If $u_2 = nu_1$ then $\frac{1}{2}(u_2^2-u_1^2) = 2u^2(n-1)/(n+1)$, where $u = \frac{1}{2}(u_1+u_2)$ is the speed of convection. When $u_2 \gg u_1$ the rate of shedding of circulation is $2u^2$. Hence we see that it is reasonable to determine the rate of shedding of circulation from the speed of convection. We put the rate of shedding of circulation equal to ku^2. It remains now to put the factor of proportionally on some foundation. We chose to make the initial calculations with parameters corresponding to a Reynolds number of about 3×10^4. At this Reynolds number we have the experimental result of Fage & Johansen (1927) at our disposal. Their experimental values of the velocities are reproduced in table 2; u_1 and u_2 are the velocities at the edges of the vortex sheet at distances x downstream from

Table 2. Velocities at the edges of the separated boundary layer.

Distances and velocities are non-dimensional, the reference distance being the cylinder radius and the velocity that of the free stream. Reynolds number = 3×10^4 (Results of Fage & Johansen 1927).

x	u_2	u_1	$\frac{1}{2}(u_2^2-u_1^2)$	$u = \frac{1}{2}(u_1+u_2)$	$\frac{\frac{1}{2}(u_2^2-u_1^2)}{u^2}$
0	1·42	0	1·01	0·71	2·0
0·408	1·37	0·07	0·94	0·72	1·81
1·0	1·38	0·23	0·93	0·80	1·45
1·592	1·25	0·35	0·72	0·80	1·12

the cylinder centre. Thus the factor k appears to have the value 1·45 at the position of our control surface. We shall reduce k from this value for two reasons. First, entrainment into the shear layer of fluid bearing vorticity of opposite sign will reduce the convection of circulation as the layer goes downstream. This effect is not automatically included in the model. We see that $\frac{1}{2}(u_2^2-u_1^2)$ decreases with x increasing in table 2 which is evidence of this effect. The reduction will be concentrated here at the point of appearance of the vortices. Secondly, the effect of the shear layer upstream of the control surface is absent in our model. The upstream vortices have the effect of reducing u below the value which we shall compute.

The rate of shedding of circulation is about $1{\cdot}0U^2$ as we see from table 2. The circulation shed from the boundary layer in one period is thus 1·6 $(2\pi Ua)$ when the Strouhal number is 0·2. The strength of laminar vortices in the wake at Reynolds numbers less than 300 is about 0·7 to $0{\cdot}8\times(2\pi Ua)$. This reduction by a factor of 2 is due to mixing of fluid separating from the two sides of the cylinder and we expect it to be to a large extent automatically included in our model. It has been shown (Gerrard 1966) that this reduction in strength is probably not strongly dependent upon Reynolds number. We shall consider here Reynolds numbers between about 10^3 and 3×10^4. In this range the vortex strength at 6 diameters behind the cylinder is about 0·5 $(2\pi Ua)$ (see Bloor & Gerrard 1966). This second reduction by a factor of $\frac{5}{8}$ is due to a process of entrainment which we do not include in our model. On this reasoning we expect the factor k to be $1{\cdot}45 \times \frac{5}{8} = 0{\cdot}91$. It remains to consider the effect on k of the vortex sheet upstream of the control surface.

From table 2 we see that the average rate of convection of circulation between $x = 0$ and 1 is 0·96. The mean speed of convection over the same distance is 0·74. Hence the circulation of the shear layer between $x = 0$ and 1 is 1·3 (Ua) or in the units of table 1 the circulation is 0·21 ($2\pi Ua$). If we lump this circulation in a point vortex at (0·5, 1·1) the x component of the velocity which it produces at the point of appearance (taken as $x, y = 1{\cdot}0, 1{\cdot}2$) is found to be $-0{\cdot}066$. From table 2 the velocity at the point of appearance is expected to be $0{\cdot}8U$ and so without the upstream vortex sheet we expect to calculate it to be $0{\cdot}87U$. On this basis k is to be reduced by a factor of $0{\cdot}84 = (0{\cdot}8/0{\cdot}87)^2$, to give $k = 0{\cdot}76$. When computations were performed with $k = 0{\cdot}75$ the circulation, on the average, contained between $x = 1$ and 2 was about 0·6 ($2\pi Ua$). With this vortex strength replacing the strength 0·21 used above we obtain a final value of $k = 0{\cdot}58$. For this reason calculations have been performed with $k = 0{\cdot}5$ and 0·75.

The actual expression used in the computation of the strength of the nascent vortex was $k\Delta t(u^2 - ZV^2)$, where Δt is the time interval of one step of the calculation. The second term contains a disposable parameter Z and the velocity at the rear of the cylinder (defined on figure 2). This second term was included to account for circulation produced at the rear surface of the body. This seems now to be less significant than the circulation swept forward from the wake but was retained in the model. The inclusion or omission of the second term is indeed found to produce little effect.

3·2·3. *The representation of scale effect*

The main representation of scale effect in this model is through the variation of the width of the near wake. We will first of all discuss the variation of the strength of the nascent vortices with Reynolds number since this was the subject of the last section at the particular Reynolds number of about 3×10^4. Bloor & Gerrard (1966) have shown that the vortex strength rises with increasing Reynolds number above 10^4. This change in strength is attributed (Gerrard 1966) to changes in turbulent entrainment of the shear layers flanking the formation region. We must therefore expect to need to change the factor k of the last section in proportion to the change in final vortex strength. Thus k at Reynolds numbers of a few thousands should be less than at 3×10^4 by a factor equal to the ratio of the final vortex strengths. This ratio depends on the turbulence of the free stream and in the case of Roshko's experiments is about 1·5 but at a very low disturbance level it is about 2·0 (see Bloor & Gerrard 1966).

It is difficult to see how the effects of the diffusion of the shear layer in the formation region can be satisfactorily lumped into the variable k. To this extent the model is unsatisfactory and so some discrepancy between the results and observation is to be expected.

The major scale effect remaining and which can be easily incorporated in the vortex model is the width of the formation region at the control surface. This width is correlated with Roshko's (1954) notched hodograph theory wake width, d'. The value of the (positive) y coordinate of the position of appearance of the elementary vortices was therefore chosen to be

$$y = \tfrac{1}{2}d' + Rv,$$

where R is at our disposal to some extent and v is the velocity component perpendicular to the free-stream direction computed for that point. The wake semiwidth, $\frac{1}{2}d'$, was determined from experimental values of the velocity outside the boundary layer at separation (Gerrard 1965). By using this expression the wake width oscillates because v oscillates: also the mean value of y is not $\frac{1}{2}d'$ because v does not oscillate about zero. The most reasonable choice of R is to equate it to the time interval of one step of the calculation. This value has been adopted usually. With R small the oscillations die out. In over-simplified models of the wake which were tried first, in which the vortices were not free but constrained to move along prescribed paths, the same result was obtained. In these cases also the oscillations increased with time if R was too large. This simpler type of model oscillates at a frequency determined solely by the feedback mechanism which as previously explained by the author (1966) is not expected to be the case in fact. Variation of the more numerous disposable parameters of the simpler models can alter the frequency considerably, by a factor of 4 or more around the Strouhal frequency.

There is an effect on v of the vortex sheets upstream of the control surface. The oscillation in v may be expected to be over emphasized. This could be counteracted by reducing R slightly but this has not been done. The effect on the mean value of v is less important. For the calculation of the force coefficients the nascent vortex is assigned only a free-streamwise component of velocity. The displacement of the mean y coordinate of the point of appearance is an effective change in Reynolds number which is taken into account.

3·2·4. *The variables in the final model and the treatment of particular effects*

We have not yet discussed the choice of time interval. It is obvious that one obtains a better representation the smaller the time interval but the computing time is long and so one is restricted in the choice of interval. Coupled with the length of computing time is the time required for the model to become independent of the initial conditions. The results have shown the adequacy of a time interval of 0·5 a/U, where a is the cylinder radius and U the free-stream speed. With this interval just sufficient computing time was available for the model to settle down and oscillate for two or three periodic times.

Two other aspects of the model need to be mentioned. When two vortices approach to a small separation each produces a large velocity at the other. Oppositely signed vortices in close proximity could remove themselves from the wake at high speed. This was avoided by removing one vortex and lumping all the vortex strength in the other when a vortex separation of less than 0·05 radii occurred. Whether or not this coalescence occurred usually made no difference to the results. In fact in many of the results presented this device was not included in the programme.

In the author's paper (1966) on the mechanics of the formation region some discussion of reversed flow close behind the cylinder was undertaken. It was found that this occurred in the computed results. When vortices are cast back against the rear surface of the cylinder it is not clear what the result will be. Two extreme cases have been investigated. On the one hand a vortex swept against the cylinder will produce circulation of opposite sign by the action of viscosity at the surface. In the extreme case the vortex could destroy itself by the production of an equal amount of opposite circulation. At the other extreme the vortex, which will tend to move along the surface, may be swept into the separating shear layer with little change in strength. The phenomena at the two extremes are represented by different procedures when a vortex recrosses the control surface. The two sets of results are sometimes significantly different.

The number of independent variables in the model has been reduced to a minimum. The main available parameters are those which vary with Reynolds number most strongly, namely, the wake width and the vortex strength factor, k. The variables describing the position at which the nascent vortices make their appearance can only be changed within a narrow range. The remaining variable is the length of time interval which will be shown to have been given a small enough value.

4. Results

With the independent variables of the model taking the values determined as described in the last section it remained to set the model in motion and observe its development. The choice of the wake width at the position of appearance of the vortices was equivalent to a choice of Reynolds number. Corresponding values of the Reynolds number were obtained by comparing the resulting properties of the flow with experimental values. The results, for this reason, are presented as corresponding to a particular Reynolds number. It must be stressed that a scale effect is only applied to the wake width and to the strength of the shear layers at their inception at the control surface. The effects of transition to turbulence in the free shear layers and of entrainment of fluid into these layers is not fully accounted for. These will contribute additionally to the scale effect. Variables corresponding to two Reynolds numbers have been investigated; one a high subcritical value, the other in the range in which the smallest values of lift have been observed in a stream free from disturbances. The resulting quantities will be compared with experimental values in §§4·1·6 and 4·2.

4·1. *Wake width corresponding to a Reynolds number of* 10^4

The basic wake semiwidth ($\frac{1}{2}d'$ in § 3·2·3) was set at a value of 1·1 radii. The final mean value of this quantity turned out to be 1·25 which corresponded to a Reynolds number of 10^4 according to Roshko's (1954) theory and our measurements of the velocity at the separation point of the boundary layer. The other main variable, k, determining the strength of the vortices, has been given the values 0·5 and 0·75: we present the results for $k = 0{\cdot}5$ first. The lift and drag coefficients are shown as functions of time in figure 1. We note that the model settles down to an almost constant amplitude oscillation in which the amplitude of the lift and drag have roughly the values observed. When the model has settled down the steady and unsteady components of the lift (see table 1) are approximately equal in amplitude but are not in phase. The phase difference alters between zero and $\frac{1}{2}\pi$. The spikes on the waveforms of figure 1 are associated with vortices cast back toward the rear of the cylinder inside the formation region. Figure 2 shows the waveform of the velocity at the rear of the cylinder. Also in this figure the times at which vortices coalesce and at which vortices recross the control surface are indicated. In the programme from which these figures are drawn the reversed flow vortices are entrained into the nascent vortices. The mean strengths of the individual vortices are about $\pm 0{\cdot}1$ $(2\pi Ua)$

and these oscillate at the first harmonic frequency with an amplitude of about 0·08. In figure 3 the velocities at the top and bottom of the cylinder are plotted. Their mean level and level of oscillation are roughly in agreement with observations at those points outside the boundary layer.

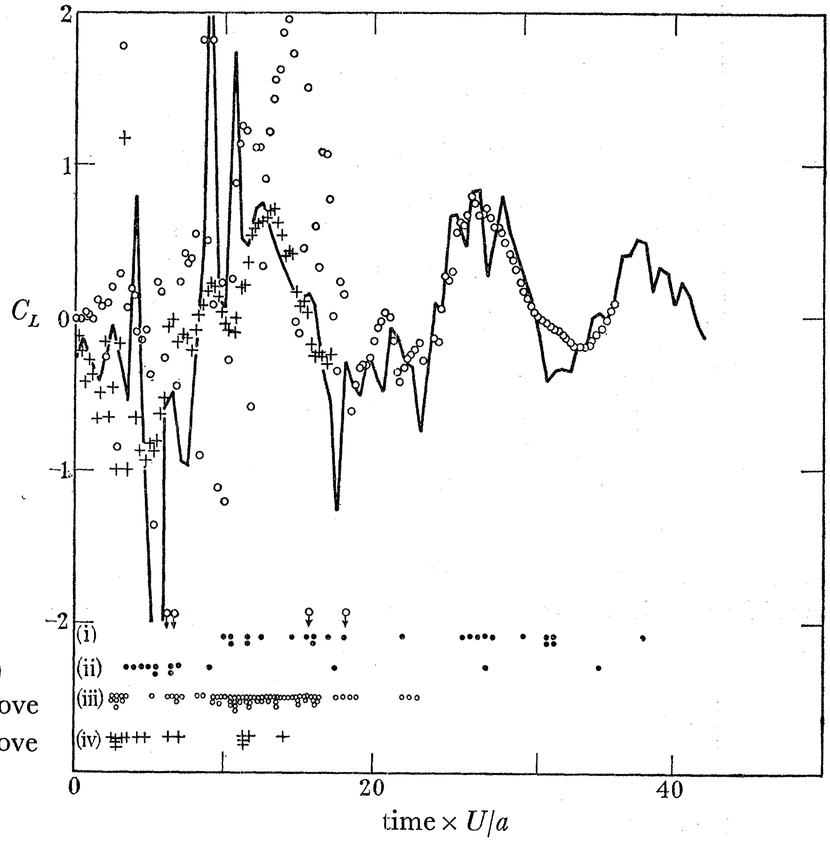
The effect of ignoring the vortex sheets upstream of the control surface must be assessed before the final values or the accuracy of the results are obtained. Figure 9 shows that there are usually two elementary vortices representing the vortex sheet between $x = 1$ and 2. We therefore look at the effect of placing a vortex of strength 0·2 half way between the shoulder of the cylinder and the point of appearance of the vortices. This produces a decrease in the velocity S (figure 3) of 0·19. The mean value of S appears from figure 3 to be 1·6. The application of the correction above gives a resulting value of the mean velocity at the shoulder of the cylinder of 1·4. This agrees well with the measured value (Gerrard 1965) at a Reynolds number of 3×10^4: the mean value of S at a Reynolds number of 10^4 is about 1·34. The effect on the force coefficients is more complicated for this depends upon the speed as well as the position of the vortices. If the vortices, above and below, are symmetrical with respect to velocity as well as position the correction to C_L is zero; the correction to C_D is $\Delta C_D = 1{\cdot}6 + 3{\cdot}7v$ when the vortices travel downstream with unit speed and outwards with speed v. If the cross stream components of velocity are the same (both upwards) at the top and bottom, $\Delta C_L = 1{\cdot}6v$ and $\Delta C_D = 1{\cdot}6$. The asymmetrical part of v is small. The error in C_L is therefore not expected to be more than 0·1. The outward velocity calculated at the point of appearance of the vortices is too large because of ignoring the vortex sheet upstream of the point of appearance. This is partly counteracted by setting this component equal to zero for the nascent vortex in the calculation of the forces. The outward speed (v) of the vortex next nearest to the body however remains. These effects are important for the calculation of C_D. The correction

$$\Delta C_D = 1{\cdot}6 + 3{\cdot}7v$$

quoted above is partly balanced by a correction for the vortices further downstream moving outward too rapidly. Vortices of strengths $\pm 0{\cdot}2$ situated at $(x, y = 1{\cdot}2, \pm 1{\cdot}2)$ gives $C_D = -0{\cdot}9 - 1{\cdot}3v$, where v is their outward speed. It is therefore possible that the correction to C_D may not be large but since it is the difference of two quantities which are each of the same order as the drag coefficient no great reliance can be placed on the mean value of C_D obtained from the model. Similarly we may suppose that the model can only be relied upon to give the order of magnitude of the oscillating part of the drag force. Despite the remarks above about the mean value of the drag, figure 1 shows an interesting feature in the drag force. There is a slow variation in mean drag. Oscillations of period an order of magnitude longer than the fundamental have been attributed (Gerrard 1966) to three-dimensional effects but these are absent here. We suggest that this is evidence of the existence of a low frequency motion in the wake. The period of revolution of the wake vortices is an order of magnitude longer than the shedding period.

4·1·1. *The effect of changing the independent variables*

In figure 4 the waveform of the lift coefficient in figure 1 is compared with waveforms obtained when several of the variables are altered. Two other waveforms are plotted in both of which the coalescence of vortices closer than 0·05 is removed. In both cases also the time interval of a step of the calculation is halved, to be 0·25 a/U. The first points, which terminate at 17 a/U, are otherwise obtained under the conditions of figure 1. We see that the reduced time interval smooths out the fluctuations considerably but otherwise produces little change. The points lie close to the figure 1 lift waveform at times greater than 11·5 a/U. The second set of data which terminates at 36 a/U closely agrees with figure 1 from 17 a/U onwards. For these points the initial vortex configuration was radically changed. The initial configuration for the last set of points have little resemblance



(i) coalescence (figure 2)
(ii) entrainment (figure 2)
(iii) entrainment for ○ above
(iv) entrainment for + above

FIGURE 4. The effect of changing the time interval of the calculations and of removing the coalescence of vortices which come close together. —, As in figure 1; +, time interval 0·25, no coalescence; ○, time interval 0·25, no coalescence and initial vortex configuration altered to an unrealistic geometry. Times of entrainment are indicated.

to the vortex distribution finally obtained in the wake. We see therefore that the model becomes independent of the initial conditions before a time of 20 a/U. The discrepancy at 32 a/U is attributed to the difference between programs with and without coalescence. This is an indication of the extent to which the model is sensitive to small disturbances produced by vortices coalescing. The entrainment of vortices which are swept backwards produces large disturbances when the vortices have appreciable strength. In these circumstances large spikes appear on the waveforms. The reversed flow vortices are entrained without change of strength for all points on figure 4. The times of entrainment are indicated.

In figure 5 the result of the alternative treatment of the reversed flow vortices is shown on the lift coefficient waveform. It is seen that the amplitude and frequency of the lift are only slightly affected but the phase is changed. There are many reversed flow vortices at the beginning of the motion. It appears that the only difference between the two extreme treatments of the reversed flow vortices is in the time taken for the model to

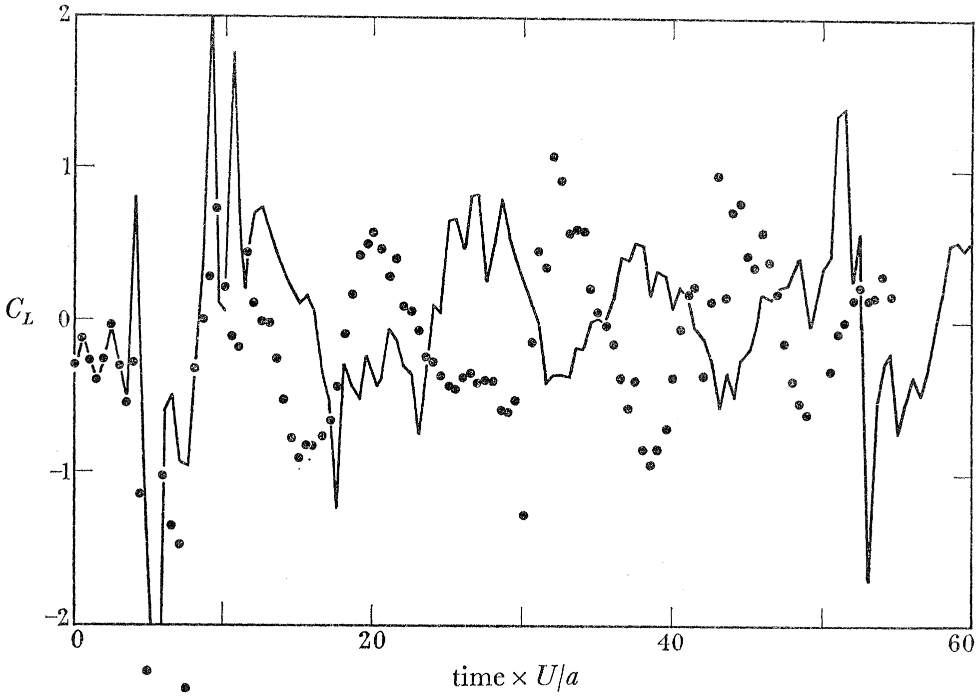


FIGURE 5. Comparison of two extreme treatments of the reversed flow vortices. —, As in figure 1, reversed flow vortices entrained without change in strength; ●, reversed flow vortices disappear.

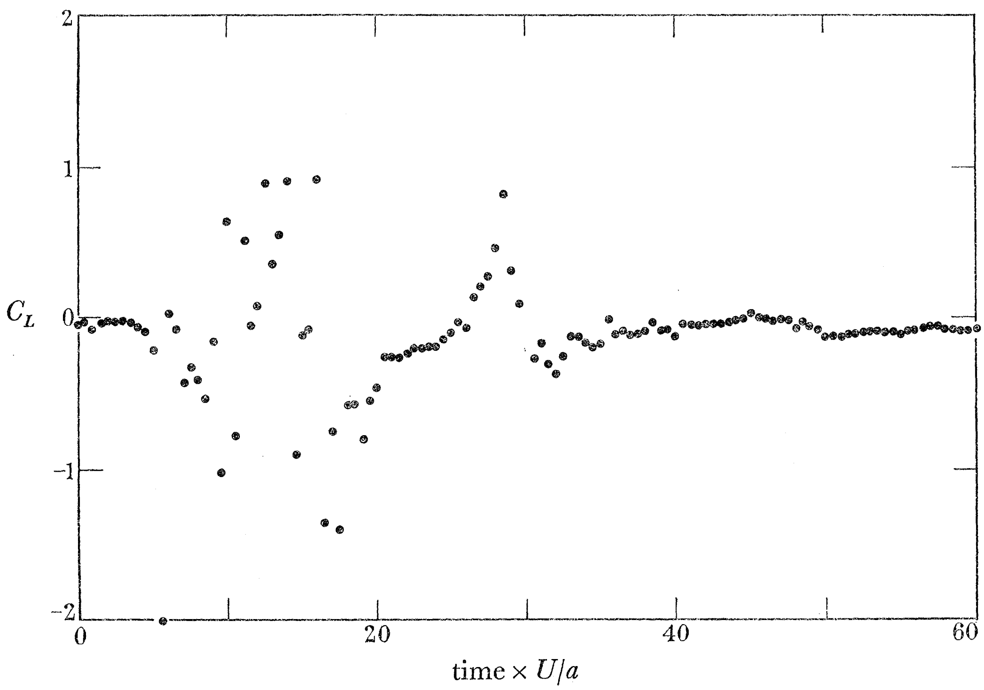


FIGURE 6. The effect of reducing R (§3·2·3), the factor governing the amplitude of oscillation of the point of appearance of the vortices. R is reduced from 0·5 to 0·25.

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become independent of the initial conditions. The mean values and amplitudes of C_D, S and T are unaltered.

The effect of the vortices which have been ignored upstream of the point of appearance is to reduce the amplitude of the oscillation of the point of appearance. The change in the parameter R (in § 3·2·3) which governs this amplitude is expected to be small. The effect of a drastic change in R is shown in figure 6. After progressing not very differently from figure 1 up to about 30 a/U the oscillation suddenly dies out. This effect will be discussed more fully in conjunction with the wide-wake, lower Reynolds number, conditions below. The change in character of the oscillations is coupled with the manner in which the vortex sheets roll up. Kronauer (1964) has also discovered that the oscillation of the point of appearance of the vortices is vital to the continuance of oscillations. For the whole time shown in figure 6 the mean value of the drag is small (about 0·1). The amplitude of the drag also dies out at about 30 a/U.

The parameter Z in § 3·2·3 was not expected to have a large effect: in the results presented so far it has been given the value 0·3. A programme exactly as that from which figure 1 was drawn was repeated upto a time 22 a/U with Z reduced to zero. Except for the details of the high frequency fluctuations the results were not significantly altered.

In §3·2·2 we came to the conclusion that the vortex strength factor k should have a value of about 0·6 .The results presented above have all been obtained with k = 0·5. The

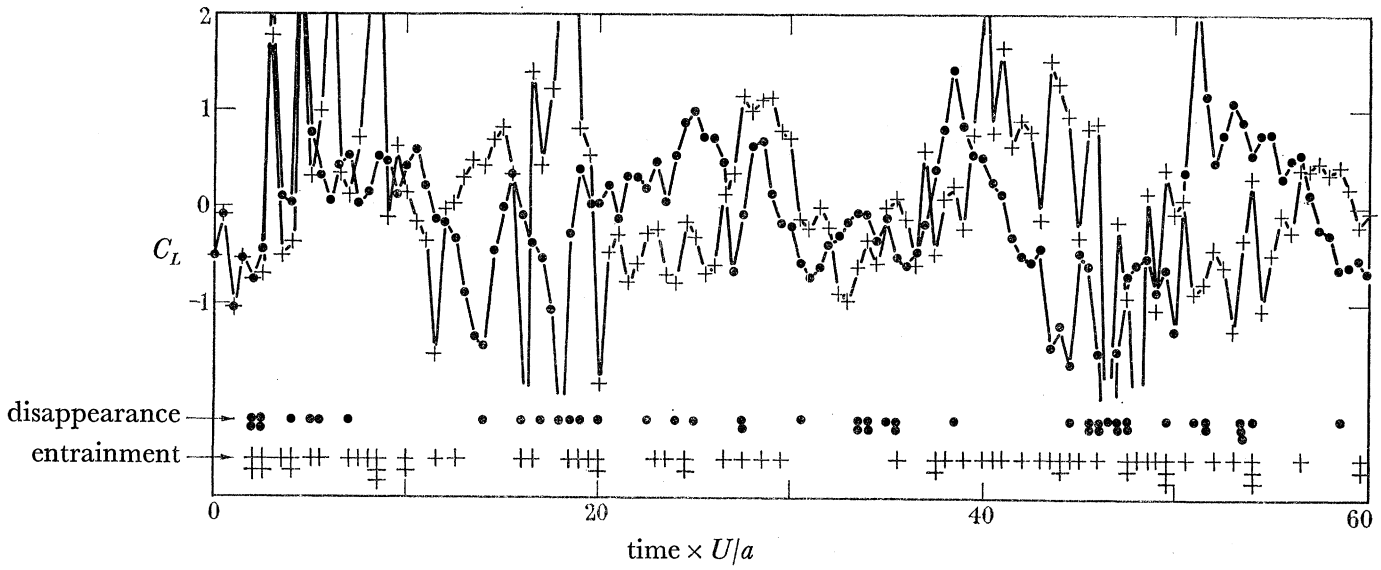

FIGURE 7. Lift coefficient wave forms for k = 0·75 showing the two extreme treatments of the reversed flow vortices. +, Reversed flow vortices entrained; ●, reversed flow vortices disappear. Times of reversed flow vortices crossing the control surface are indicated.

effect of making k = 0·75 was investigated. The effect of Reynolds number change is to alter the wake width as well as k. For the analysis of the scale effect it is instructive to look at the effect of changing k and the wake width separately.

In figure 7 the waveforms of the lift coefficient are shown for k = 0·75. Again the two extreme treatments of the reversed flow vortices are shown. The greater high frequency fluctuation is immediately obvious; coupled with this is the more continuous reversed flow of vortices at the rear of the cylinder. In general the results are less satisfactory than those presented above with $k = 0.5$. The main point at which the results are unsatisfactory is the internal coherence of the model when $k = 0.75$. The mean wake semiwidth with the stronger vortices is increased to about $1.4a$. At this much larger width we expect k to be smaller by as much as 50 % from the value we argued it should have at a Reynolds number of 3×10^4 in §3·2·2.

The other results which accompany figure 7 are not drastically different from those presented at the lower value of k. The mean drag coefficient is about unity but the increased high frequency fluctuations mask the first harmonic oscillations of the drag. The mean values of S and T are about 1·2 but when corrected for the effect of vortices closer to the body than the control surface a value less than 1·0 would be obtained which is not in agreement with experiment. The amplitude of S and T similarly are in greater disagreement with observations, being almost 50 %. The frequency is on the whole lower by as much as 50 %, though one period still retains the highest frequency observed in figures 1 to 3. It is not legitimate to use these values to interpolate between the results with $k = 0.5$ and 0·75 because of the lack of consistency within itself of the model at $k = 0.75$. We therefore take the results obtained with $k = 0.5$ as being the best representation. It is unfortunate that calculations were not made with the variables arranged so that the resulting mean wake width was the same as obtained with $k = 0.5$ but for a larger value of k. We may expect that the results for $k = 0.5$ will not be greatly in error.

4·1·2. *The frequency*

We see from figures 1 to 3 that the frequency is not sharply defined. There are periods corresponding to a Strouhal number ($S = Nd/u$) of 0·20 but others which correspond to lower frequencies. It is not possible to generate enough periods of oscillation to obtain an accurate frequency determination. The mean Strouhal number over the time interval from 30 to 60 a/U is 0·185. The author's paper (1966) on the mechanics of the formation region leads us to expect that only one of the two length scales which determine the frequency will be automatically included in our model. The diffusion effect in the shear layer is omitted. We expect, from the previous work mentioned above, the frequency, N, to be given by

$$\frac{Nd}{U} = 0.2 \frac{d}{l_f} \frac{d}{L}.$$

We shall see in §4·1·4 that this model generates a formation region length of 1 to 1·5 diameters: the experimental value is 1·2 diameters thus with $Nd/U = 0.185$ and this value of l_f/d we find that L/d should be 0·9. This is in agreement with the values expected for this quantity at Reynolds numbers between 3 and 4×10^4 from the authors previous work (1966).

4·1·3. *The wake*

The computer programs provide the positions of all the vortices at any time specified. Figure 8 has been constructed from two sets of such positions at times 23 and 24 a/U under the conditions with which figure 1 was obtained. In the lower diagram of figure 8 filament lines have been drawn through the vortices on the principle that filament lines cannot cross. We ignore the fact that all the vortices from one side do not pass through a point but originate over a small region of the control surface.

The first, though obvious, observation is that the points become concentrated into a vortex street configuration. The vortices are not very concentrated, as Albernathy & Kronauer (1962) have already shown. One may describe the vortices as possessing a tail

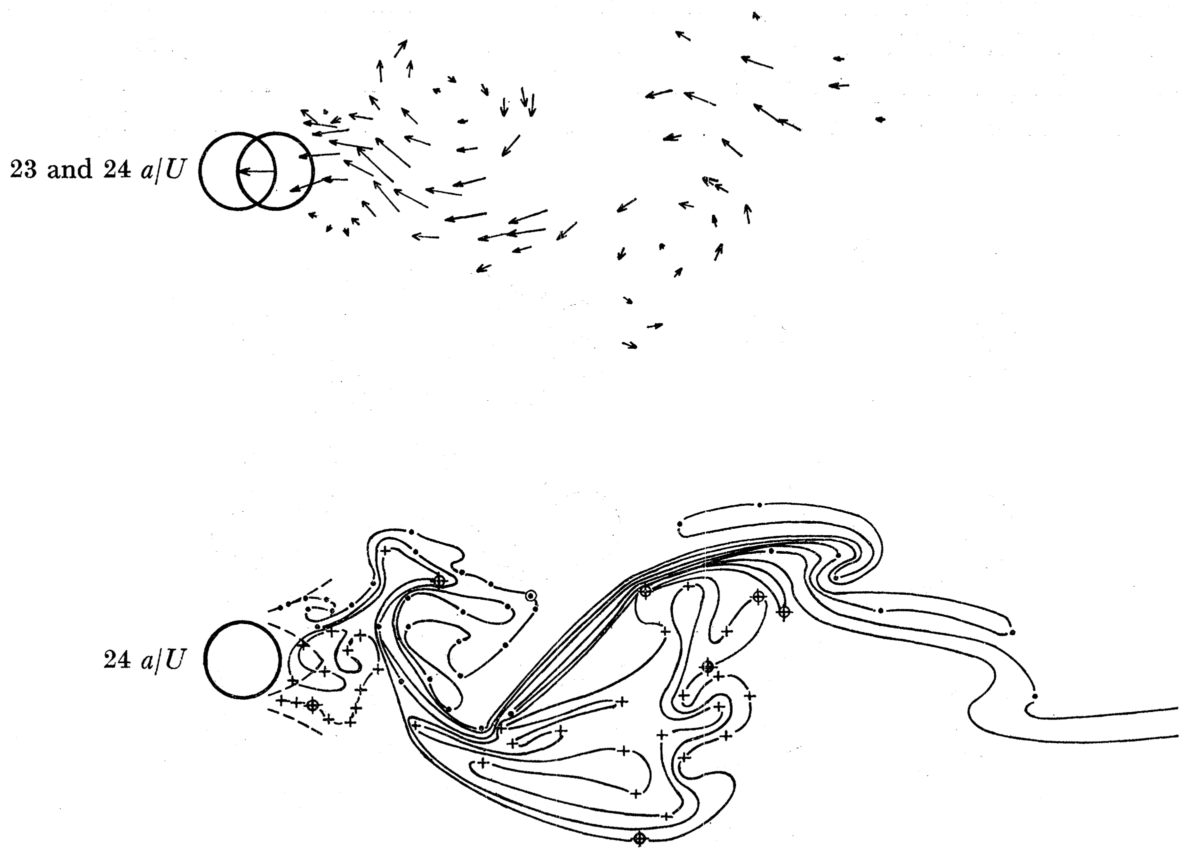

FIGURE 8. Filament line diagram and incremental particle paths at time 24 a/U under the conditions of figure 1.

of vortices trailing behind the main concentration. Vortices of opposite sign are often found in this tail. The circled points in figure 8 are vortices which due to coalescence have changed their signs.

The upper diagram of figure 8 shows an alternative representation obtained by joining the positions of the vortices at times 23 and 24 a/U by arrows directed towards the positions at the later time. The main effect of the alternative portrayal is the appearance of greater concentration of vorticity. This is concerned with the already well known danger in the interpretation of filament line flow visualization pointed out by Hama (1962). The picture in figure 8 is as seen by an observer moving with the free-stream speed not with the speed of the vortices.

The dashed lines just behind the cylinder in figure 8 are reproduced from Goldstein's figure 227 (1938): they represent the positions of the mean shear layers at a Reynolds number of 14480.

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4·1·4. *The formation region*

There remain other results which may be compared with experiment. These concern the formation and strength of the vortices in the wake. We examine first the formation region. The results presented here served to shape the physical discussion undertaken in an earlier paper (1966). The development of the flow close behind the cylinder for the conditions of figure 1 is shown by filament lines in figure 9. The diagrams give a complete description of one half cycle from time 20 to 30 a/U. It may be seen from figure 1 that this cycle is a long one compared with those following this time interval. At times 20 and 27 to 29 a/U we see the vortex sheet drawn across the wake cutting off the growing vortex

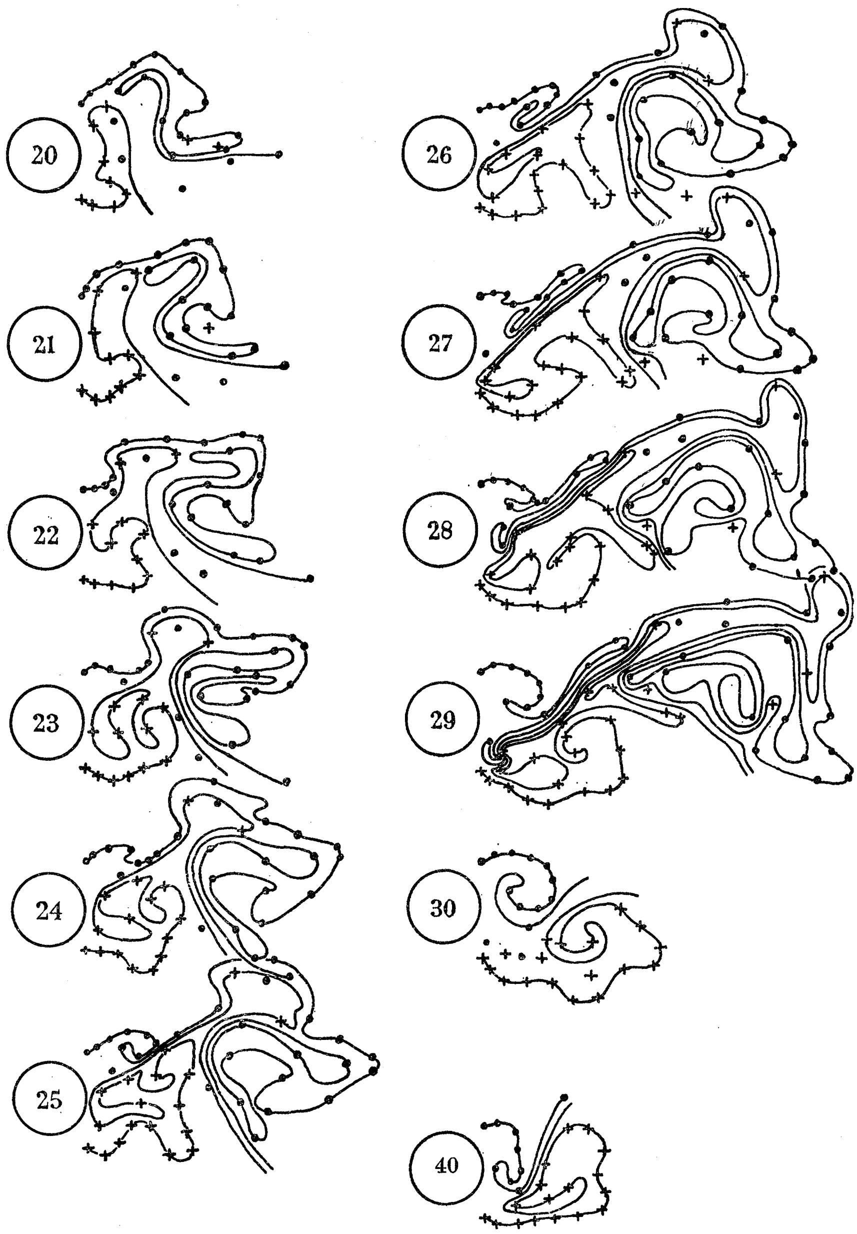

FIGURE 9. The formation region development as a function of time shown by filament lines. Conditions as in figure 1.

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from a further supply of circulation. We see also that the growing vortex is stationary until it is shed by this process of drawing across the other vortex sheet. The downstream speed of the vortex when shed may be calculated from figure 9 to be 0·55 U: This low value of convection speed is attributed to its connection with the fact that this half period is a longer one than the average in figure 1.

Some of the points in figure 9 are not connected by filament lines. These vortices are associated with those out of the picture which were shed at earlier times. One sees one of these involved in reversed flow across the control surface between times 27 and 28 a/U.

The length of the formation region is difficult to determine exactly from figure 9. The formation region (Bloor 1964) terminates where irrotational flow is first drawn across the axis of the wake. This is occurring in figure 9 from the bottom at about 1·1 diameters downstream of the centre of the cylinder at time 20 to 22 a/U; from the top at about 1·2 diameters downstream at 29 a/U; and again from the top at about 1·0 diameter downstream at 40 a/U.

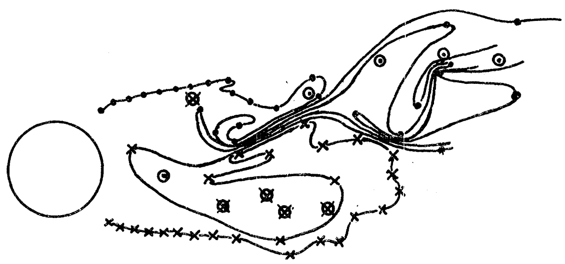


FIGURE 10. The formation region shown by filament lines at time 45 a/U when the oscillation had ceased in figure 6.

The formation region was quite different when the oscillation was made to die out by reducing the oscillation at the point of appearance of the vortices (see figure 6). One map of the formation region in this condition is shown in figure 10. The vortex sheets continue more independently in figure 10. Though the sheet from the other side is drawn across the wake it does not cut the vortex off. This behaviour will be considered in more detail below in connexion with the model with a wider wake which behaves similarly in certain conditions.

4·1·5. *The strength of the vortices in the wake*

The determination of the strengths of the vortices in the wake is not simple because the construction of boundaries is largely a matter of choice when the circulation is spread over large areas. All experimental determinations of vortex strength in the wake rely on matching the observed velocity field to a model of the wake. Such models employ circularly symmetrical vortices and the fact that a satisfactory matching is obtained does not preclude the possibility of just as good a match with a very different vorticity distribution.

In general the vortex formation we have observed with this model has a tendency towards the production of more than one vortex from each side per period. This confirms the findings of Abernathy & Kronauer (1962). The forming vortex is shed by drawing across circulation from the other side of the wake. This process has the effect of producing

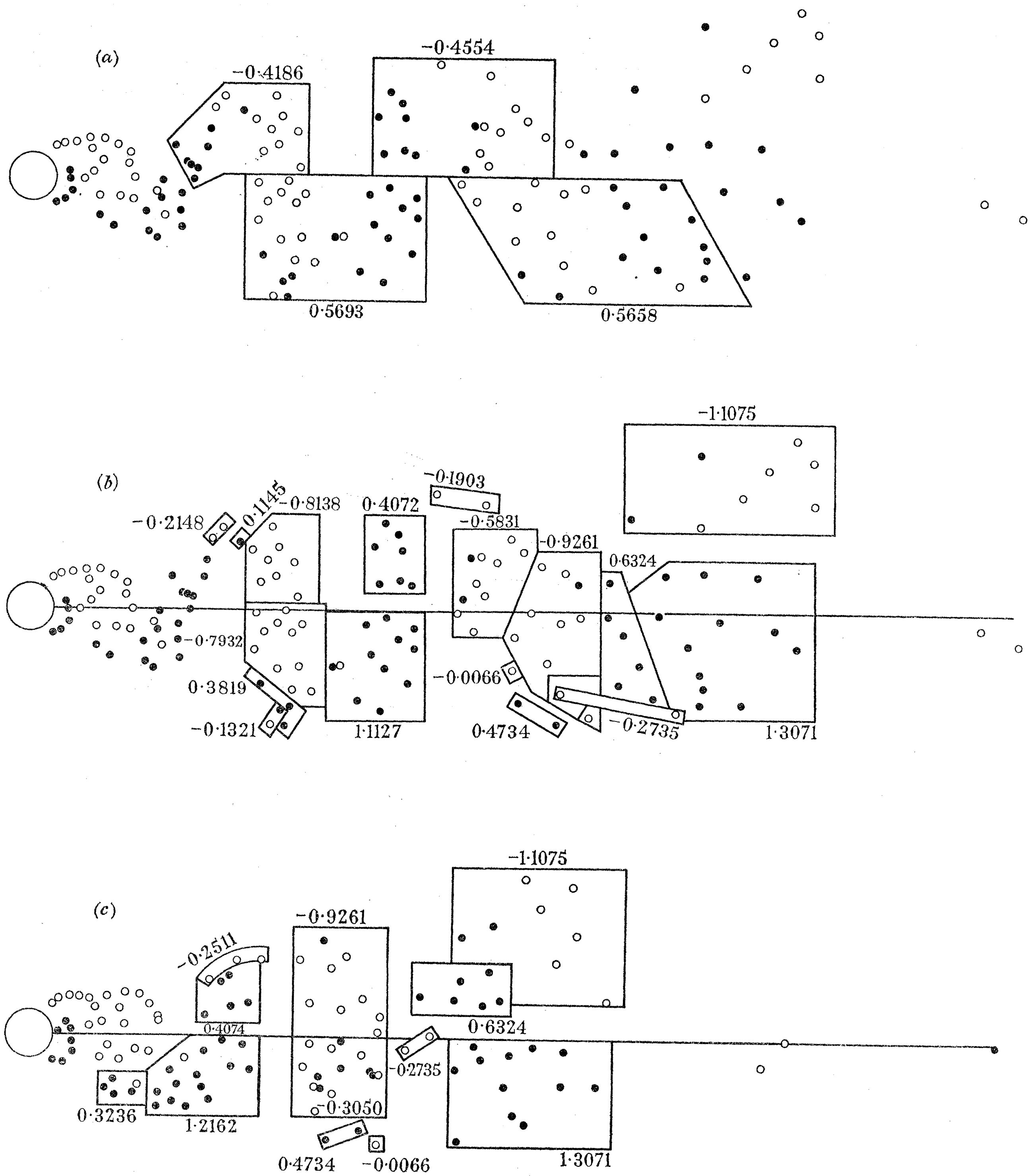


FIGURE 11. (a) Vortex positions at 45 a/U. Conditions as in figure 1. The wake is divided to obtain vortices of strength agreeing with observations (Bloor & Gerrard 1966). The numbers on the figure are vortex strength in units of $2\pi Ua$. (b) Alternative division of the wake. (c) The same wake at 35 a/U.

a tail to the original vortex with some oppositely signed circulation between the body and tail of the vortex. The details of such processes are likely to be sensitive to the presence of viscosity (or eddy viscosity).

The downstream development of the wake was investigated for the conditions under which figure 1 was obtained. Figure 11 (*a*) shows a division of the vortices in the wake which agrees reasonably with the values of vortex strength obtained from measurements (Bloor & Gerrard 1966). In figure 11 (*b*) an alternative division of the wake is made. One sees that an entirely different impression of the vortex strengths may be given. Figure 11 (*c*) shows the same wake at an earlier time. We see that in this time interval of 10 a/U the vortices have travelled downstream almost one wavelength whereas the components of the vortices have only rotated about their centre of vorticity by something like one-eighth of a revolution. The longer time scale, it was suggested above, is responsible for the slow variations observed in figure 1.

4·1·6. *Comparison of the results of the computation and observations*

In table 3 the results of the computations which have been discussed above are collected together. The results are presented in terms of the Reynolds number at which the corresponding observed quantity is found. The experimental values are taken from data to which reference has already been made.

Table 3

quantity	value computed	corresponding Reynolds number of observed quantity
mean wake semiwidth	basic 1·1	3×10^4
	resulting 1·25	1×10^4
corrected, C_L amplitude	0·7 to 0·8	2×10^4
Strouhal number and hence	0·185	—
diffusion length, L/d	0·9	$3{\cdot}5\times10^4$
mean velocity, S	1·4	3×10^4
amplitude of S	16 %	maximum observed 7 %
vortex strength	*ca.* 0·5	4×10^4
length of formation region	2·2	3×10^4

We conclude that the model produces values in agreement with all the quantities which we are able to compare with observation at high Reynolds number. The main shortcoming of the model is in the omission of certain of the effects of viscosity and transition to turbulence. This produces only small discrepancies at this Reynolds number which are the same as we expected from our earlier work on the subject (Gerrard 1966).

4·2. *The results obtained with maximum wake width*

The wake width varies with Reynolds number and has a maximum value at a Reynolds number of about 2000. This maximum value depends upon the disturbance level of the free stream. We look now at the result of taking a wake width equal to the maximum obtained at a low disturbance level which is larger than that inferred from measurements in a turbulent stream. We take a basic wake semiwidth of 1·4. The final mean wake semiwidth was 1·6. Here we expect, for low disturbance level, that the value of the vortex strength parameter, k, should be one half of the value found to apply at the high sub-critical Reynolds numbers (see §3.2.3). We expect therefore that k should be about 0·3. Several values of k have been investigated.

Figure 12 shows the resulting lift coefficient at values of k of 0·5 and 0·25. We see a very large difference in both the lift amplitude and frequency. The results obtained with $k = 0{\cdot}25$ are as expected for the measured lift coefficient is 0·02. It is difficult to make

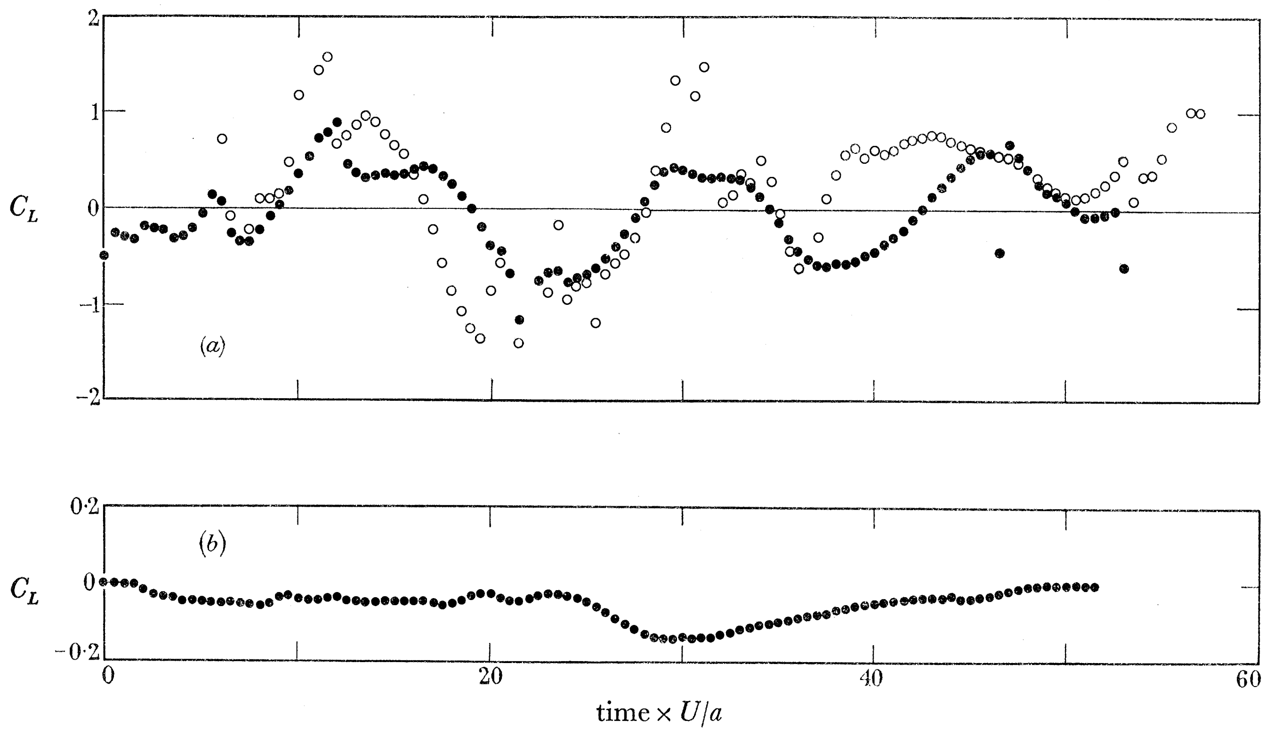


FIGURE 12. Lift coefficient waveforms with a wide wake. Reynolds number 2000; low disturbance level. $R = 0{\cdot}5$, $Z = 0$, time interval 0·5. (a) ○, $k = 0{\cdot}5$, reverse flow vortices entrained; ●, $k = 0{\cdot}5$, reverse flow vortices disappear; (b) ●, $k = 0{\cdot}25$, reverse flow vortices disappear.

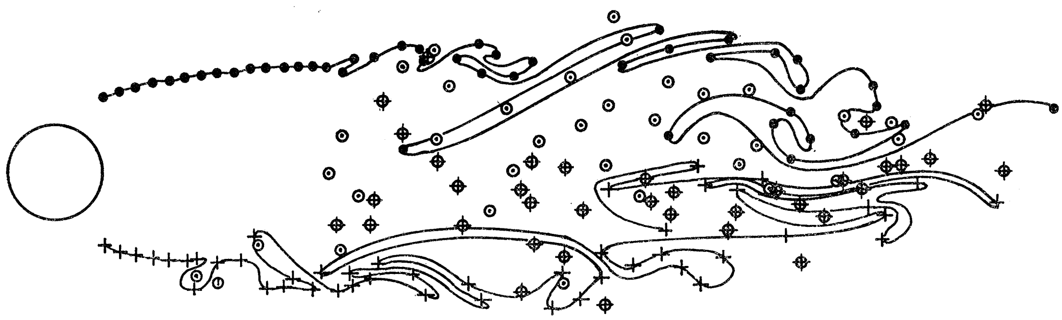


FIGURE 13. Symmetrical formation region at time 45 a/U for $k = 0{\cdot}25$ case of figure 12.

more than an order of magnitude comparison at these low values, especially when the frequency is not clearly defined. The spectacular dependence of lift amplitude upon the vortex strength parameter k is a reflexion of the way in which the formation region develops. When $k = 0{\cdot}5$ the formation region is not significantly different from that obtained with a narrower wake except in scale. When $k = 0{\cdot}25$, on the other hand, a much more symmetrical formation region develops as figure 13 shows. This is not in agreement with the observation that the length of the formation region is about 5 radii.

There is evidence that the frequency is of the expected order of magnitude. If we ignore the effect of the diffusion length we expect $Nd/U = 0{\cdot}2 \times d/l_f$. If we take for l_f, the experimentally observed value of $2{\cdot}5d$, the resulting non-dimensional period is $25\,a/U$. There is slight evidence that this period is present in the low amplitude wave of figure 12.

In order to gain insight into the development of the symmetrical formation region we consider results obtained with $k = 0{\cdot}4$ in which a transition from the oscillating to relatively non-oscillating state has taken place. Figure 14 shows the lift waveform under these conditions. The times of disappearance of the vortices recrossing the control surface

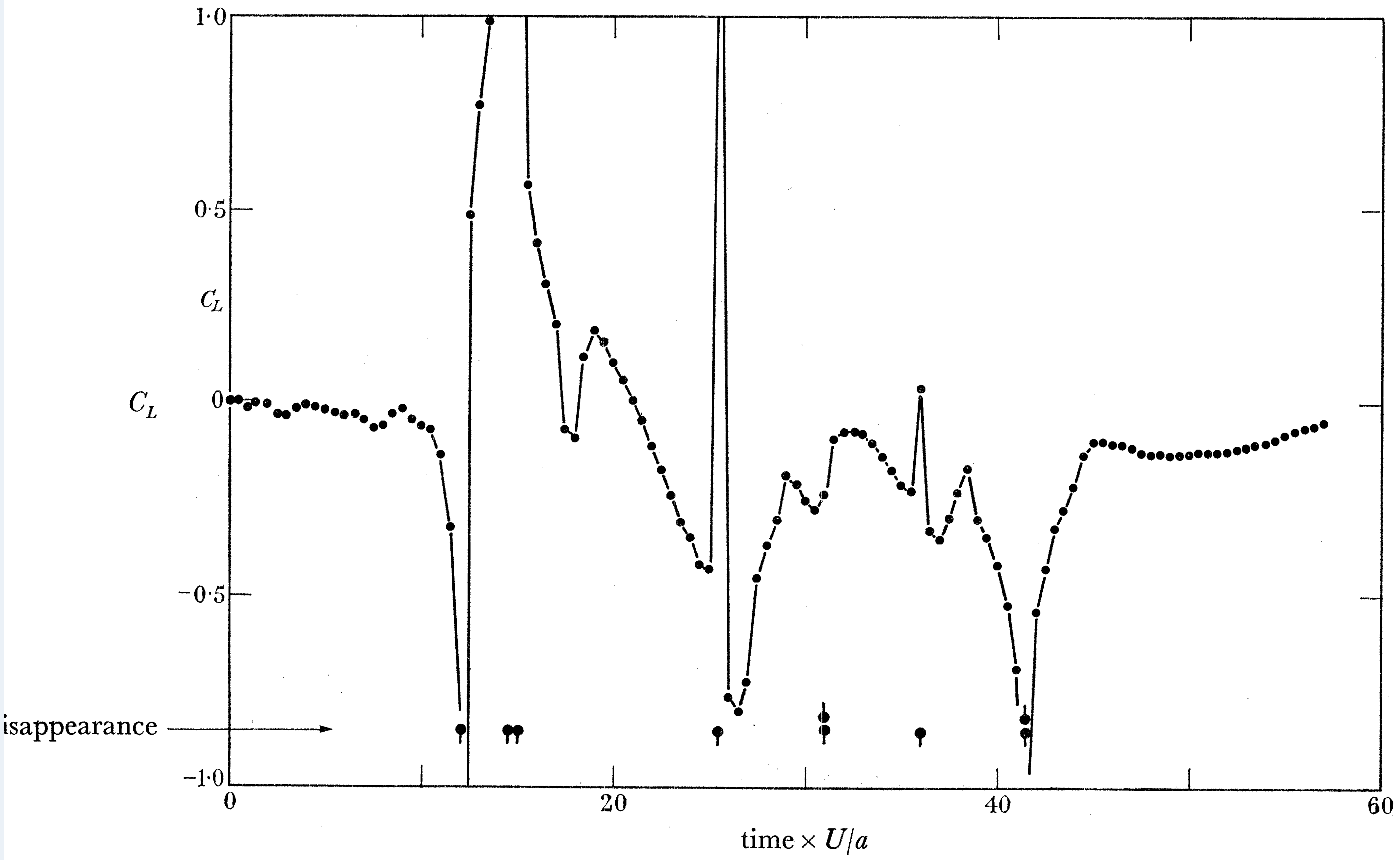
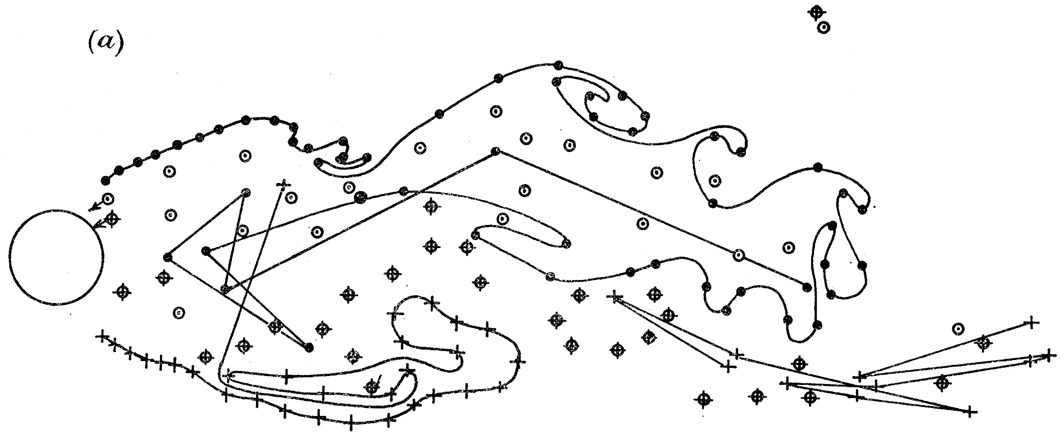


FIGURE 14. Lift coefficient waveform showing transition from the oscillating state to one of considerably smaller amplitude. $k = 0{\cdot}4$, $R = 0{\cdot}5$, $Z = 0$, time interval $= 0{\cdot}5$. Times of disappearance of the reverse flow vortices are shown.

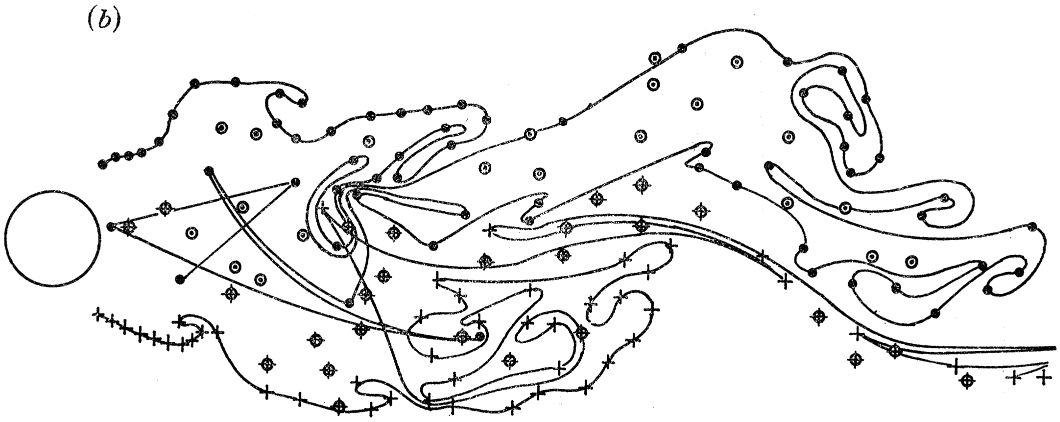
are also shown. It is difficult to say which is cause and which is effect in this situation. When the formation region is asymmetrical, reversed flow vortices are observed and an appreciable oscillating lift is present. On the other hand disturbances of some sort are necessary to produce an oscillation as we saw when the movement of the point of appearance of the vortices was inhibited. Herein probably lies the reason why a disturbance free stream is necessary before the very small lift values are observed.

The transition from the oscillating to the more quiescent flow is illustrated in figure 15 by means of filament line diagrams of the formation region. All the vortices close to the

(a)



(b)



(c)

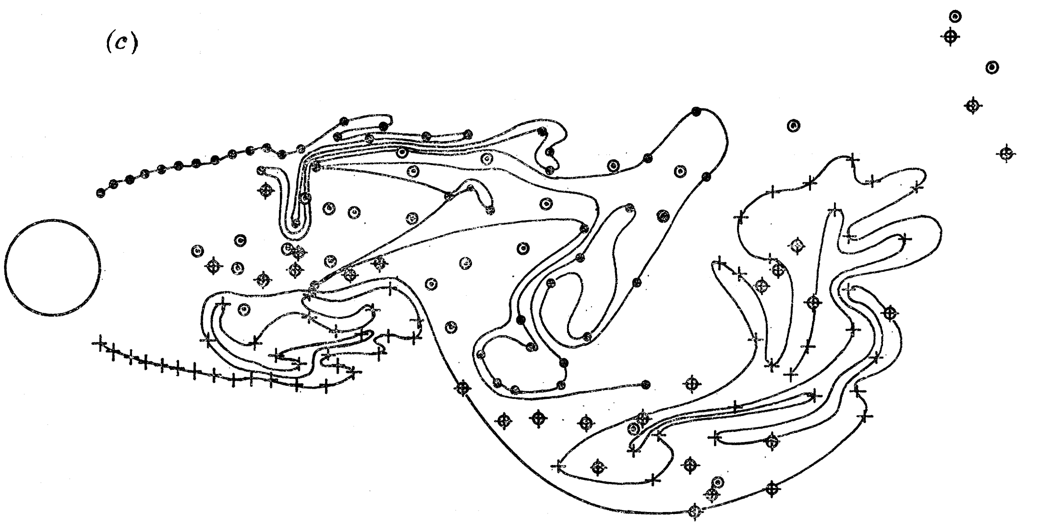


FIGURE 15. Vortex positions at different times during the decay of oscillations in figure 14. (a) At 40 a/U; (b) at 45 a/U; (c) at 55 a/U. Circled vortices are older than those through which the filament lines are drawn.

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cylinder are indicated but only the filament lines through the vortices most recently shed are drawn. The older vortices are circled to distinguish them. Figure 15 (*a*) shows the pattern at 40 a/U when the oscillation is present. We see a concentration of positive vorticity at the bottom beginning to draw across the opposite shear layer at $x = 5$ to 7. The vortices are not as concentrated as they are in figure 9. The growing vortex is not stationary as it was in figure 9. We see from figure 15 (*b*) that the process of growing of a vortex, the drawing across the wake of the shear layer on the opposite side and the growing of the next vortices are all taking place together. Even by the time 45 a/U there is no striking dissimilarity between the formation region here and at the high Reynolds number. The difference is one of scale rather than of type. By the time 55 a/U, however, when the oscillation has ceased there is a striking difference which is shown in figure 15 (*c*). Between downstream distances of 4 and 8 the patterns at 40 a/U and 55 a/U are similar. Close to the body there is a marked symmetry in the vortex sheets. The vortices downstream of 8 radii are less separated than previously. This we presume is a minor effect. The basic difference is in the near wake where there are no reversed flow vortices and where the vortex sheets are symmetrical. The symmetry exists in the speeds as well as the positions of the vortices. For this symmetrical arrangement to develop we suggest that two factors must be present; the wide wake which allows separate development of the vortex sheets and the absence of disturbances in the form of free-stream turbulence or body vibration.

It is not known whether the diminutive lift force measurements were of a continuous low amplitude waveform or an intermittent waveform. It is possible that when there is a probe in the wake the diminutive amplitude oscillations are impossible. This would explain why a formation region length of about 2·5 diameters is measured with a hotwire (Bloor 1964).

There seems no reason to analyse the results as minutely as we did at the higher Reynolds number. The smallness of the amplitudes of the oscillating quantities, when other disturbances are present, precludes accurate determination in a time of one or two cycles.

5. Conclusions

A vortex model of the wake has been used to determine the oscillating variables of the flow past a circular cylinder. The model was based on a physical foundation but omitted the modelling of transition to turbulence and the effects of viscosity in the wake. Previous work by the author (1966) allowed an estimate of the effect of these omissions to be made. Within these limitations the model generates values in reasonable agreement with all the oscillating quantities available for comparison with experiment. This extended comparison was restricted to a single high subcritical Reynolds number. The scale effect is incorporated in the model by means of values of the wake width at inception, determined from Roshko's (1954) theory and also by means of values of the strength of the vortex sheets determined from experimental results and a physical discussion of the flow.

The other flow investigated was that at a Reynolds number of about 2000, in the absence of disturbances. In this flow the main observation for which an explanation was lacking was the extremely small values of certain oscillating quantities. The model generated these diminutive oscillations when the independent variables were given the values that the physical discussion indicated.

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The essential characteristics of the model are contained in the determination of the rate of shedding of circulation from the mean velocity calculated at a point of appearance of the elementary vortices. This point is removed downstream from the boundary layer separation points. The point of appearance is only to some extent at our disposal for the effect of the vortex sheets upstream of it must be determined and used to correct the results. The control surface, on which the points of appearance lie, must on the otherhand not be too near the end of the formation region. Because of the effect of vorticity upstream of the control surface the error in the calculation of the drag force may be large and is not accurately determinable. It was found that the oscillation of the points of appearance is essential to the continuation of oscillations.

An extension of the method employed here may involve the inclusion of a modelling of transition to turbulence and entrainment by the turbulent shear layers. The method here is more simple minded than a computer solution of the equations of motion but has, for this reason, served to analyse the effect of different characteristics of the flow.

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